QUANTUM BOLTZMANN EQUATIONS FOR MIXING SCALAR FIELDS

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We report on a work in progress, whose goal is a systematic field theoretical derivation of the quantum transport equations for baryon production in the electroweak plasma at a first order phase transition in the limit of slowly varying background fields (thick wall limit). We start with the Schwinger-Dyson equations for the two point Green function written in the closed time contour (CTC) formalism. The quantum Boltzmann equations for the density matrix arise when the SD-equations are expanded to the first order in the gradients in the on-shell limit. In this paper we consider only scalar fields, but the formalism can easily be extended to fermions.

1 Introduction

The matter-antimatter asymmetry to be explained is most accurately pinned down by nucleosynthesis considerations. Allowing for a fair uncertainty in the present observational situation, one has [1]

$$\eta_B \equiv \frac{n_B}{s} \simeq 2 - 7 \times 10^{-11}. \tag{1}$$

An exciting realisation is that η_B could have been generated at the electroweak phase transition. What is required is a strong transition and an adequate source of CP-violation. While the minimal standard model (MSM) is known to fail these conditions [2], they may well be met in its popular minimal supersymmetric extension (MSSM) [3]. There are three distinct parts of the problem: (1) *equilibrium calculations*, which provide information about the strength of the phase transition [2]; (2) *phase transition dynamics*, whose study provides the phase boundary shape and speed [4]; and finally (3) *baryon production computation*. The goal of our program is to bring problem (3) to the level of understanding of (1).

At the moment there is no systematic quantum field theoretical derivation of the CP-violating sources and the transport equations that would consistently describe all aspects of the problem in one formalism. The difficulty lies in the inherent out-of-equlibrium nature of the problem. The theory we study here,

is defined by the following scalar lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \phi^\dagger m^2(x) \phi - \mathcal{L}_{\rm int}\,, \tag{2}$$

where $\mathcal{L}_{\rm int}$ contains the interaction terms. The field ϕ can have several components so that m^2 is in general a complex matrix varying in space and time, possibly giving rise to out-of-equilibrium conditions and nontrivial CP-properties, which are the necessary ingredients for baryogenesis.

2 The Kadanoff-Baym equations

We start our analysis with the exact Schwinger-Dyson equations in the Keldysh closed time contour (CTC) formalism [5,6,7]. The basic quantity is the out-of-equilibrium 2-point Green function

$$G^{\alpha\beta}_{\mathcal{C}}(x,y) = -i \left\langle T_{\mathcal{C}} \left[\phi_\alpha(x) \phi^\dagger_\beta(y) \right] \right\rangle \tag{3}$$

where $T_{\mathcal{C}}$ defines time ordering along the contour $\mathcal{C}$ which starts at some t_0 (often taken to be at $-\infty$) goes to $+\infty$, and then back to t_0. From now on we suppress the indices α, β, which denote different scalar fields ("flavours"), and work with matrices in this flavour space. The two point function $G_{\mathcal{C}}(x,y)$ obeys the contour Schwinger-Dyson equation

$$G_{\mathcal{C}}(x,y) = G^0_{\mathcal{C}}(x,y) + \int_{\mathcal{C}} dx' \int_{\mathcal{C}} dx''\; G^0_{\mathcal{C}}(x,x') \Sigma_{\mathcal{C}}(x',x'') G_{\mathcal{C}}(x'',y)\,, \tag{4}$$

where $\Sigma_{\mathcal{C}}$ is the self-energy and $G^0_{\mathcal{C}}$ is the free particle (tree-level) propagator. In order to solve $G_{\mathcal{C}}(x,y)$ from (4), some additional information about $\Sigma_{\mathcal{C}}$ must be provided. In general $\Sigma_{\mathcal{C}}$ is specified through the usual BBGKY hierarchy of equations for n-point functions. In the weak coupling limit $\Sigma_{\mathcal{C}}$ can be computed perturbatively, if the interaction lagrangian $\mathcal{L}_I$ in Eqn. (2) is given; for the moment however, we can keep $\Sigma_{\mathcal{C}}$ completely general.

After some algebra Eqn. (4) can be recast as the well-known *Kadanoff-Baym* (KB) equations [8]:

$$\begin{aligned} (G_0^{-1} - \Sigma^{r,a}) \otimes G^{r,a} &= \delta \\ (G_0^{-1} - \Sigma^{r}) \otimes G^{<,>} &= \Sigma^{<,>} \otimes G^{a}. \end{aligned} \tag{5}$$

where $\otimes$ denotes the convolution integral: $[f \otimes g](x,y) \equiv \int dx' f(x,x') g(x',y)$ and $G^{r,a}$ are the retarded and advanced Green functions and

$$G^{>}(x,y) = -i \langle \phi(x) \phi^\dagger(y) \rangle\,, \qquad G^{<}(x,y) = -i \langle \phi^\dagger(y) \phi(x) \rangle \tag{6}$$

are the quantum Wigner functions. The Kadanoff-Baym equations (5) are still exact, but formidable to solve. They considerably simplify when expanded to the first order in gradients.

3 Gradient expansion

Gradient expansion is valid in the limit of slowly varying background fields. This means that the Compton wave length $\ell \sim k^{-1}$ of excitations should be smaller than the characteristic length scale of the background variation. More precisely, propagators and self-energies should vary slowly with respect to the macroscopic (average) coordinate $X = (x + y)/2$, *i.e.*

$$|G| \gg |\partial_X \partial_k G| \,, \qquad |\Sigma| \gg |\partial_X \partial_k \Sigma| \,, \tag{7}$$

where the precise meaning of momentum k is defined below. Applying this *gradient expansion* to Eqns. (5) we get the new KB-equations

$$e^{-i\diamond}\{G_0^{-1} - \Sigma^{r,a}\}\{G^{r,a}\} = 1 \tag{8}$$

$$e^{-i\diamond}\{G_0^{-1} - \Sigma^{r}\}\{G^{<,>}\} = e^{-i\diamond}\{\Sigma^{<,>}\}\{G^{a}\} \,, \tag{9}$$

where the Green functions $G = G(k; X)$ and self-energies $\Sigma = \Sigma(k; X)$ are now defined in the mixed representation, defined by the Fourier transform with respect to the relative coordinates $r = x - y$ (Wigner transform):

$$G(k; X) = \int d^4 r e^{ik\cdot r} G(X + r/2, X - r/2) \,. \tag{10}$$

The $\diamond$-operator of Eqn. (9) is defined as the following generalization of the Poisson brackets:

$$\diamond\{f\}\{g\} = \frac{1}{2}\left[\partial_X f \cdot \partial_k g - \partial_k f \cdot \partial_X g\right] . \tag{11}$$

To the first order in gradients, $e^{-i\diamond}$ in Eqn. (9) simplifies to $1 - i\diamond$. For the electroweak phase boundary the typical macroscopic scale is of the order of the boundary thickness $L \sim (10-20)/T$[9,4], whereas for a typical plasma excitation $\partial_k \sim k^{-1} \sim T^{-1}$, so that $\partial_X \partial_k \sim 1/LT \ll 1$, *i.e.* criteria (7) are satisfied.

To the lowest order in gradients equations (8–9) clearly define the familiar propagators of an excitation in a translationally invariant background, and their poles define the well known quasiparticle dispersion relations. Thus Eqn. (8) in particular contains information about the spectrum of excitations of the system, and it is contained in the spectral function defined by

$$\mathcal{A} \equiv \frac{i}{2}\left(G^r - G^a\right) = \frac{i}{2}\left(G^> - G^<\right) . \tag{12}$$

4 Propagator equation

We begin by considering the propagator equation for the retarded and advanced Green functions. For the one-field case Eqns. (8) imply

$$\cos\diamond\{G_0^{-1}-\Sigma^{r,a}\}\{G^{r,a}\}=1 \tag{13}$$

$$\sin\diamond\{G_0^{-1}-\Sigma^{r,a}\}\{G^{r,a}\}=0\,. \tag{14}$$

Now observe that Eqn. (14) can be obtained from Eqn. (13) by an application of the differential operator $\tan\diamond$, and hence it gives no new information. Therefore, to the first order in gradients, we have simply

$$G^{r,a}=(G_0^{-1}-\Sigma^{r,a})^{-1}. \tag{15}$$

From this, one can easily read off the spectral function $\mathcal{A}$ (12) and $G_R = (G^r+G^a)/2$.

In the case of several flavours Eqn. (15) does not solve the propagator equation (8) to the first order in gradients. For simplicity consider $\Sigma_R \equiv (\Sigma^r+\Sigma^a)/2 \to 0$. Then the explicit solution has the following form

$$G^{r,a}=\mathcal{U}^\dagger\frac{\det[\lambda]}{\det[\lambda]\,\lambda+k_z\big[\big[H_z,\lambda\big]_-,\lambda\big]_- \pm i\epsilon\omega(\det[\lambda]+\lambda\mathrm{tr}[\lambda])}\mathcal{U}\,, \tag{16}$$

where $\mathcal{U}$ is the unitary matrix in the flavour space that diagonalizes the mass matrix m^2, $H_z=-i\mathcal{U}\partial_z\mathcal{U}^\dagger$ is a "gauge" field, and $\lambda=\mathcal{U}(G_0^{-1}-\Sigma_R)\mathcal{U}^\dagger$ is the diagonal matrix with the entries $\lambda_\pm=\omega^2-\omega_\pm^2$, $\omega_\pm^2=\vec{k}^2+m_\pm^2$, and $m_\pm^2$ are the diagonal entries of $m_D^2=\mathcal{U}m^2\mathcal{U}^\dagger$ for the two field case. Note that an inhomogeneous background alters the pole structure of the propagator. Indeed, Eqn. (16) can be diagonalized, and one can show that each of the homogeneous particle poles $\omega_\pm$ is split into three poles, each with a weight 1/3 of the original excitation, which lie on the circle in the complex frequency plane, with shifts given by $\{\delta\omega,\delta\omega e^{2i\pi/3},\delta\omega e^{-2i\pi/3}\}$, where $\delta\omega\propto(H_z)_{12}^{2/3}$ is a nonanalytic shift. One can show, however that these nonanalytic shifts affect the on-shell Boltzmann equation at the second order in gradients and hence can be neglected [10]. Finally we note that it is not hard to generalize Eqn. (16) and include the self-energy Σ_R.

5 On-shell approximation for the Quantum Bolzmann Equation

The hermitean part of Eqn. (9) defines the quantum Boltzmann equation (QBE), and anti-hermitean part a nondynamical constraint equation (CE).

The QBE is the *dynamical master equation* whose solutions specify how the effective phase space of the system is populated, and the CE is the non-dynamical constraint which singles out the physical solutions. In the on-shell limit the QBE (upper signs) and OSE (9) simplify to

$$\begin{aligned} &-i\diamond\{G_0^{-1}-\Sigma_R\}\{G^<\}_\mp+\{G_0^{-1}-\Sigma_R\}\{G^<\}_\mp \\ &=\tfrac{1}{2}\Big(\{\Sigma^>\}\{G^<\}_\pm-\{\Sigma^<\}\{G^>\}_\pm\Big)+\{\Sigma^<\}\{G_R\}_\mp \end{aligned} \tag{17}$$

where $\{a\}\{b\}_\mp = ab \mp ba$. In the following we only consider the QBE.

Eqns. (17) specify the dynamics of the quantum Wigner function $G^<$. In the on-shell limit the singular structure of $G^<$ can in general be factored out as follows

$$iG^< = \mathcal{A}_s n + n^\dagger \mathcal{A}_s = \pi \mathrm{sign}(\omega) \sum \tilde{g}^i \delta(\omega-\omega_i)\,, \qquad \text{for} \quad \Gamma \to 0\,, \tag{18}$$

where n stands for the (regular) matrix of "occupation numbers."

Upon integrating the QBE from Eqn. (17) over positive frequencies, we obtain the following *on-shell* quantum Boltzmann equation for particles:

$$\partial_t f + \frac{1}{2}\{\partial_{\vec{k}}\tilde{\omega}_d, \partial_{\vec{x}} f_d\} - \frac{1}{2}\{\partial_{\vec{x}}\tilde{\omega}_d, \partial_{\vec{k}} f_d\} + \frac{i}{\hbar}[\tilde{\omega}_d, f_d] + Rot\ f_d = Coll[f_d]. \tag{19}$$

This is our main result. Analogous equation for antiparticles is obtained by integrating over negative frequencies. Here $\tilde{\omega}_d$ is the (diagonal) matrix of the quasiparticle frequencies. The derivative terms, including the anticommutators, form the hermitian many-field generalization of the usual Boltzmann flow term, and the commutator term is the expected mixing term familiar from the usual Liouville equation. This term is the source of the nontrivial quantum coherence effects, and together with some additional rotational terms included in *Rot* f it leads to potential new sources for baryogenesis. The collision term $Coll[f_d]$ in (19) depends on the detailed form of the interactions. In addition to the usual diagonal collision integrals it contains new terms which tend to damp the off-diagonal elements of f_d to zero.

Due to the nonlocal character of the SD equation, f_d is related to $G^{<,>}$ in a nontrivial manner in that there is a derivative correction:

$$f_d = \frac{1}{2\hbar}\{\tilde{\omega}_d, \tilde{g}_d\} - \frac{i}{4\hbar}\left[\partial_{\vec{k}}\tilde{\omega}_d, \partial_{\vec{x}}\tilde{g}_d\right] + \frac{i}{4\hbar}\left[\partial_{\vec{x}}\tilde{\omega}_d, \partial_{\vec{k}}\tilde{g}_d\right], \tag{20}$$

where $\tilde{g}_d = \mathcal{U}\tilde{g}\mathcal{U}^\dagger$ is defined in Eqn. (18). This definition is the unique one that renders the QBE in the simple form of Eqn. (19).

The basic structure of Eqn. (19) is basis independent; when rotated into another basis by an unitary matrix $\mathcal{U}$, f_d and $\tilde{\omega}_d$ become $f = \mathcal{U}^\dagger f_d \mathcal{U}$ and $\tilde{\omega} = \mathcal{U}^\dagger \tilde{\omega}_d \mathcal{U}$, respectively, where $\tilde{\omega}$ in general is not diagonal. Further, the detailed form of $\mathcal{R}ot$ term changes. Physically the $\mathcal{R}ot$ term arises as a consequence of the space-time dependent rotation required to bring an excitation to the local WKB basis; discussion of its detailed form is beyond the scope of this talk.

In the adiabatic limit, where the wall width L is much larger than the interaction rates, $L \gg 1/\Gamma$, collisions efficiently damp away the off-diagonal elements (unless interactions are flavour blind), resulting in quantum decoherence. In this case the quantum terms vanish, and one obtains the diagonal semiclassical Boltzmann equations for the quasiparticle distribution functions, which were studied earlier in refs. [11] and [9], with only the classical force $\vec{F}_{cl} = -\partial_{\vec{x}}\omega_k$ and "spontaneous" terms to fuel baryogenesis. However, we conclude by noting that even when $\Gamma L \gg 1$ is satisfied, it may be important to include the non-adiabatic effects, when we are interested in phenomena that vanish in the adiabatic limit, as may be the case with CP violating effects; the result is then suppressed by $(\Gamma L)^{-1}$.

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References

1. K. Olive, hep-ph/9901231.
2. K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov *Nucl. Phys.* **B466** (1996) 189.
3. See the talks by J. Cline hep-ph/9902328 and M. Quiros in these proceedings.
4. G. Moore and T. Prokopec, *Phys. Rev.* **D52** (1995) 7182.
5. P. Danielewicz, *Ann. Phys.* **152** (239) 1984; G.D. Mahan, *Phys. Rep.* **145** (251) 1987; J. Rammer and H.J. Smith *Rev. Mod. Phys.* **58** (1986) 323.
6. S. Mrowczynski and U. Heinz, *Ann. Phys.* **229** (1994) 1.
7. P. Henning, *Phys. Rep.* **253** (1995) 235.
8. L.P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, Benjamin Press, New York (1962).
9. J.M. Cline, M. Joyce and K. Kainulainen, *Phys. Lett.* **B417** (1998) 79.
10. M. Joyce, K. Kainulainen and T. Prokopec, hep-ph/9906411.
11. M. Joyce, T. Prokopec and N. Turok, *Phys. Rev. Lett.* **75** (1695) 1995; *Phys. Rev.* **D53** (2958) 1996.